8.4 - Matrix Exponential

Recall that $e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ This is the power series representation for the exponential function. In the same way, we can define a matrix exponential.

Definition: For any $n \times n$ matrix $\mathbf{A}$,

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

Example: Compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$.

$\mathrm{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Consider $\mathbf{X}' = \mathbf{A}\mathbf{X}$. If $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$, then $\mathbf{X}' = \dfrac{d}{dt}\left(e^{\mathbf{A}t}\mathbf{C}\right)$. To find $\dfrac{d}{dt}\left(e^{\mathbf{A}t}\right)$, we use the definition:

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^2}{3!} + \cdots\right)$$

$\mathbf{X}' = \mathbf{A}e^{\mathbf{A}t}\mathbf{C}$

So for $\mathbf{X}' = \mathbf{A}\mathbf{X}$, with $\mathbf{A}$ containing constant entries, $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution. (Note: $\mathbf{C}$ is a column matrix of arbitrary coefficients.)

Example: Use the matrix exponential to find the general solution of the given system.

$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{X}$

Initial-value problems

$e^{\mathbf{A}t} = \mathbf{\Phi}$ is the fundamental matrix for the system, so variation of parameters yields, the general solution

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t} \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds$$

Note: $e^{-\mathbf{A}s}$ is $e^{\mathbf{A}t}$ with t replaced by $-s$. In our work, we will take $t_0 = 0$.

Example: Find the general solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Example: Solve the given system by diagonalizing the coefficient matrix.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix} \mathbf{X}$$